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Note

Split-Step Spectral Method for Nonlinear Schrödinger Equation with Constant Background Intensities

In 1973 Hasegawa and Tappert [1] showed that inclusion of an intensity-dependent refractive index in the evolution equation describing pulse envelope propagation in an ideal lossless optical single-mode fiber leads to the nonlinear Schrödinger equation (NLSE). Zakharov and Shabat [2, 3] have shown analytically that the NLSE has soliton solutions irrespective of the fiber group-velocity dispersion (GVD). In the case of anomalous dispersion, GVD < 0, the solitons are termed bright as they consist of a light pulse. On the other hand, only "dark" solitons which consist of a localized dip in a constant background intensity exist, in the case of normal dispersion, GVD > 0. In 1980 bright solitons were observed experimentally [4], while an experimental indication of dark soliton propagation has been obtained only recently [5, 6].

The NLSE with periodic boundary conditions, has been solved numerically by the split-step spectral method described in [7]. In an earlier note [8], the method was improved by including an additional term in the NLSE, which absorbs the outgoing radiation at the boundaries (see also [9]). However this term was only capable of absorbing radiation in the case of bright soliton evolution. In this note we generalize the method to include dark pulse propagation. Dark pulses have previously been treated numerically by Blow and Doran [10], using explicit (simple) finite difference approximations. However, dark pulses appear not to have been investigated by spectral methods with absorbing boundaries.

In the appropriate system of normalized coordinates the NLSE can be written [4]

$$iu_t + \frac{\alpha}{2} u_{xx} + |u|^2 u = 0. \tag{1}$$

The quantity α assumes the value -1 for positive GVD and $+1$ for negative GVD. Without loss of generality, we may, in the case of dark pulse propagation, choose the normalized background intensity to be equal to 1, such that

$$|u(x, t)| \to 1, \qquad u_x(x, t) \to 0 \qquad \text{as} \quad |x| \to \infty \ (\text{GVD} > 0). \tag{2a}$$

For the bright soliton, we have

$$|u(x, t)| \to 0, \qquad u_x(x, t) \to 0 \qquad \text{as} \quad |x| \to \infty \ (\text{GVD} < 0). \tag{2b}$$

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In our modified version of the NLSE, we add the loss term, $i\gamma(x)\,u(|u|^2-1)$, and obtain

$$iu_t+\frac{\alpha}{2}u_{xx}+|u|^2\,u+i\gamma u(|u|^2-1)=0, \tag{3a}$$

where γ is a real function of x. Here

$$\gamma(x)=\gamma_0\{\mathrm{sech}^2\,[\beta(x-L/2)]+\mathrm{sech}^2\,[\beta(x+L/2)]\}. \tag{3b}$$

This introduces a smooth loss (gain) at the boundaries $x=-L/2$ and $x=L/2$ for $\gamma_0(|u|^2-1)>0$ (<0). The constant parameters γ_0 and β in Eq. (3b), determine the height and the width of the "barriers." The constant β is chosen such that the reflections from the boundaries are minimized, ensuring at the same time, a minimal perturbation region in between. The choice of γ_0 is dictated by the wrap-around of radiation. At the boundaries, we assume the periodicity conditions

$$u(-L/2,\,t)=u(L/2,\,t)$$

and

$$u_x(-L/2,\,t)=u_x(L/2,\,t). \tag{3c}$$

For dark pulse propagation we choose $\gamma_0>0$, and the additional term in Eq. (3a) has the effect of absorbing outgoing radiation without violating the periodicity requirement Eq. (3c) of the split-step method (i.e., no wrap-around of radiation). For bright pulse propagation, a negative γ_0 must be chosen.

To apply the generalized split-step method, the modified NLSE Eq. (3a) is separated into a linear and a nonlinear part. The nonlinear part

$$i\tilde{u}_t+|\tilde{u}|^2\,\tilde{u}+i\gamma\tilde{u}(|\tilde{u}|^2-1)=0, \tag{4}$$

has the solution

$$\begin{aligned}\tilde{u}(x,\,t)&=\tilde{u}(x,0)\\ &\times\exp\left\{i\left(t+\frac{1}{2\gamma}\ln[|\tilde{u}(x,0)|^2+(1-|\tilde{u}(x,0)|^2)\exp\{-2\gamma t\}]\right)\right\}\\ &\times\exp\left\{\gamma t-\frac{1}{2}\ln[(1-|\tilde{u}(x,0)|^2)+|\tilde{u}(x,0)|^2\exp\{2\gamma t\}]\right\}.\end{aligned} \tag{5}$$

Second, the linear part of Eq. (3a), $i\tilde{u}_t+(\alpha/2)\,\tilde{u}_{xx}=0$, is solved in Fourier space by

$$\tilde{U}(k,\,t)=\tilde{U}(k,0)\exp\{-i\alpha k^2t/2\}. \tag{6}$$

The solution is advanced one time step Δt by (i) obtaining $\tilde{u}(x,\,\Delta t)$ from $u(x,0)$

by means of (5) with $\tilde{u}(x,0)=u(x,0)$, (ii) inserting the Fourier transform of $\tilde{u}(x,\Delta t)$ as $\tilde{U}(k,0)$ in (6)

$$\tilde{U}(k,\Delta t)=\int_{-\infty}^{\infty}\tilde{u}(x,\Delta t)\exp\{ikx\}\,dx\cdot\exp\{-i\alpha k^2\,\Delta t/2\}, \tag{7}$$

and (iii) transforming the result $\tilde{U}(k,\Delta t)$ back to x-space,

$$\tilde{\tilde{u}}(x,\Delta t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{U}(k,\Delta t)\exp\{-ikx\}\,dk. \tag{8}$$

The computed value is then used in Eq. (5) instead of $\tilde{u}(x,0)$ to advance the solution another step. This method is accurate to first order in Δt and all orders in Δx and is unconditionally stable [11].

Figure 1 shows the evolution of a symmetric dark pulse

$$u(x,0)=(1-\operatorname{sech}^2 x)^{1/2}, \tag{9}$$

in the case of normal dispersion $\alpha=1$. The difference between Fig. 1a and Fig. 1b demonstrates the importance of adding absorption at the boundaries in the NLSE, in the case of pulses with nonvanishing background intensity.

Figure 2 depicts the evolution for the initial excitation

$$u(x,0)=[1-\operatorname{sech}^2(x/8)]^{1/2}, \tag{10}$$

which by a scaling is seen to correspond to $\phi(s,0)=8[1-\operatorname{sech}^2 s]^{1/2}$, a dark pulse in a larger background intensity. Note that the ordinate in Fig. 2 is $1-|u|$, providing a better visual resolution of the dark pulses.

In conclusion, it has been shown that the addition of a loss (gain) term

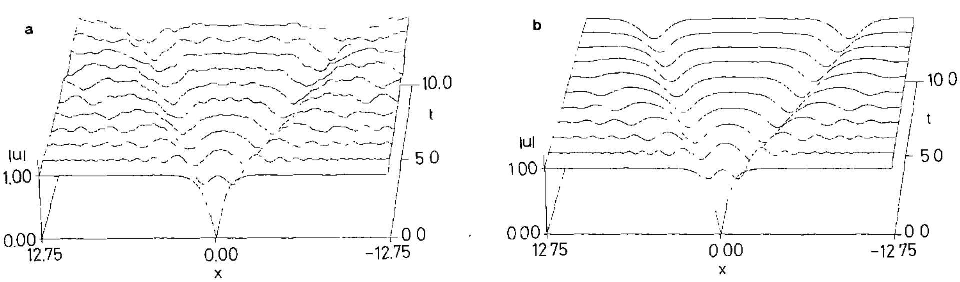

FIG. 1. Evolution of the symmetric initial dark pulse, Eq. (9), with dynamics given by Eq. (3) with normal dispersion ($\alpha=-1$) and (a) without absorption, $\gamma_0=0$, (b) with absorption, $\beta=1$, $\gamma_0=20$. $L=25.6$, $\Delta x=0.1$, $\Delta t=0.005$.

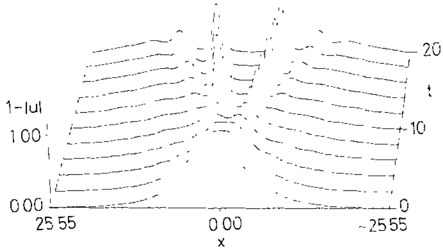

FIG. 2. Evolution of the symmetric initial dark pulse, Eq. (10), with normal dispersion ($\alpha = -1$) and absorption, $\beta = 1$, $\gamma_0 = 20$. $L = 51.2$, $\Delta x = 0.1$, $\Delta t = 0.005$.

$i\gamma(x)\, u(|u|^2 - 1)$ to the NLSE, has the effect of absorbing radiation at the boundaries in the split-step spectral method. This method is applicable to both bright and dark solitons.

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